Computers and Mathematics with Applications 62 (2011) 2314–2329

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

An M/G/1 retrial G-queue with non-exhaustive random vacations and an unreliable server

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ARTICLE INFO

Article history:
Received 7 April 2010
Received in revised form 9 July 2011
Accepted 10 July 2011

Keywords:
Retrial queues
G-queues
Random vacations
Stability condition
Markov chain
Reliability

ABSTRACT

This paper deals with an M/G/1 retrial queue with negative customers and non-exhaustive random vacations subject to the server breakdowns and repairs. Arrivals of both positive customers and negative customers are two independent Poisson processes. A breakdown at the busy server is represented by the arrival of a negative customer which causes the customer being in service to be lost. The server takes a vacation of random length after an exponential time when the server is up. We develop a new method to discuss the stable condition by finding absorb distribution and using the stable condition of a classical M/G/1 queue. By applying the supplementary variable method, we obtain the steady-state solutions for both queueing measures and reliability quantities. Moreover, we investigate the stochastic decomposition law. We also analyse the busy period of the system. Some special cases of interest are discussed and some known results have been derived. Finally, an application to cellular mobile networks is provided and the effects of various parameters on the system performance are analysed numerically.

1. Introduction

Retrial queues are characterized by the feature that arriving customers who find all servers busy or down or on vacation will join the retrial group to try their luck again some time later. Retrial queues have been widely used to model many problems in telephone switching systems, telecommunication networks and computer systems for competing to gain service from a central processing unit. For a review of main results and methods the reader is referred to the survey papers by Falin [1], Artalejo [2,3], Yang and Templeton [4]. Most retrial queueing systems assume that the time between successive repeated attempts are exponentially distributed with rate $n\delta$, when the number of customers in the orbit is n. However, recent applications to communication protocols and local area networks show that there are queueing situations in which the retrial rate is independent of the number of customers in the orbit. This retrial policy is called the constant retrial policy which is a useful device for modelling the retrial phenomenon in communication and computer networks where repeated attempts are made by processor units independently of the number of messages stored in each node of the network. The first work on this constant retrial policy is due to Fayolle [5] who assumed that only the customer at the head of the orbit queue can request for service after an exponentially distributed retrial time with constant rate of repeated attempts. Since Fayolle [5], there has been a quick growth in the literature on retrial queues with constant rate of repeated attempts; see for instance Artalejo [6], Li and Zhao [7], Kumar [8], Atencia [9] and Gomez-Corral [10].

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doi:10.1016/j.camwa.2011.07.018

Queues with negative arrivals, called G-queues, were first introduced by Gelenbe [11] in 1989 with a view to modelling neural networks. The positive customers enter a queue and receive service as ordinary queueing network customers. A negative customer will vanish if it arrives to the queue when the server is idle or down. Negative customers cannot accumulate in a queue and do not receive service. The negative arrivals affect the queue behaviour in a variety of ways. For example, (i) arrival of a negative customer which removes all the customers in the system (RCA); (ii) arrival of a negative customer which removes only a customer from the head of the system (RCH), including the customer being in service; (iii) arrival of a negative customer which removes only a customer from the end of the system (RCE). Queueing systems and networks with negative customers have many applications in telecommunication, computer networks and manufacturing systems. The negative customers can represent additional behaviours such as breakdowns, killing signals, call losses and load balancing. For example, in computer networks, if a virus enters a node, one or more files may be infected, and the system manager may have to go through a number of backup to recover the infected files. Another example is the multiple resource systems with requests for service (positive customers) and decisions to cancel a request (negative customers). Queueing systems and networks with negative customers have proved interesting theoretically and led to several publications describing their mathematical properties. For a comprehensive analysis of queueing systems with negative arrivals, the reader may refer to papers [12–22] and their references.

More recently, there has been an increasing interest in queueing systems and networks with server failures and repairs. Because in practice we often meet cases where the servers may fail and can be repaired. Besides, server breakdowns are considered as the most natural cause of service interruptions. Queueing systems with server breakdowns are very common in computers, manufacturing systems and communication networks. Queueing systems with a repairable service station have been studied by many authors. Retrial queues that take into account server failures and repairs were introduced by Aissani [23] and Kulkarni and Choi [24]. Wang et al. [25] studied a repairable $M/G/1$ retrial queueing model from the viewpoint of reliability for the first time, both the queueing indices and reliability characteristics are obtained. However, little work has been done on queues taking into account server breakdown which is represented by the arrival of a negative customer which causes some customers to be lost since Harrison et al. [26] presented a new technique for modelling unreliable queueing models based on the notion of queues with negative customers. In fact, reliability modelling using queues with negative customers has important applications. For example, in communication networks, messages are transmitted in a packet-switching mode and the server may fail during a transmission and lose one part of a message. Another example is in telephone networks, a "failure" also can be interpreted as the loss of a call.

One more feature which has been widely studied in queueing systems is vacation. Most of the analysis for the retrial queue concerns the exhaustive service schedule. Queueing systems with non-exhaustive have received little attention. In practice we often meet the case that the server starts a vacation randomly. Non-exhaustive service means that the server may start a vacation when some customers are still in the system. Various authors have analysed queueing problems of server vacations with several combinations. A literature survey on queueing systems with server vacations can be found in [27–29].

Many papers have been devoted to obtaining stability conditions for the single server retrial queues. For the $M/G/1$ retrial queues operating under the classical retrial policy, it is well known that the stability condition is that the arrival rate λ times the expected service time $\frac{1}{\mu}$ is smaller than one, and for the $M/G/1$ retrial queues operating under the constant retrial policy, the stability condition is $\lambda\left(\frac{1}{\mu}+\frac{1}{\lambda+\delta}\right) < 1$, where δ is the retrial rate. (See [1].) Some publications on ergodicity of retrial queues can be found in [30,31,9,10]. Most of authors analyse the stability conditions of their queueing models by using Foster's criterion [32] and Kaplan's condition [33] to study the embedded Markov chain at departure completion epochs. However, when the considered queueing systems are very complicated, this method is very difficult to be implemented. We present a new method based on the "generalized service time" to obtain the stability condition by finding absorb distribution and using the stable condition of a classical $M/G/1$ system. It is proved that our method is efficient computationally and is tractable for most complicated queueing systems.

In this paper, we consider a single-server retrial queue with negative customers and non-exhaustive random vacations subject to the server breakdowns and repairs which is motivated by the performance analysis of cellular mobile networks. A breakdown at the server is represented by the arrival of a negative customer which causes the customer being in service to be lost when the server is busy. The server start a vacation after an exponentially distributed time when the server is up. The principal purpose of our paper is to study the necessary and sufficient condition for the system to be stable and realize an extensive analysis of the system from both the queueing and reliability points of view.

The remainder of this paper is organized as follows. The model description is given in Section 2. The stability condition is analysed in Section 3. The steady-state distribution of the server state and the orbit length are discussed in Section 4 along with some performance measures. The stochastic decomposition property is investigated in Section 5. Section 6 relates to the reliability results obtained for this model. Section 7 focuses on the busy period of the system. Some special cases are considered in Section 8. An application for the model under discussion and some numerical examples are shown in Section 9. Finally, Section 10 concludes the paper.

2. Description of the queueing system

In this section, we consider a single server retrial queueing system in which the arrivals of both positive and negative customers are two independent Poisson processes, with rates λ^+ and λ^-, respectively. We assume that there is no waiting space and therefore, if an arriving positive customer finds the server idle, the customer starts his service immediately. Otherwise, the customer will join a group of unsatisfied customers i.e. orbit to seek the service again and again till he finds the server idle. It is assumed that the retrial times for any repeated customer are exponentially distributed with rate δ/n, given that there are n customers in orbit. The arrival of a negative customer which does not receive service not only removes the customer being in service to go out of the system, but also causes the server breakdown. At a negative arrival epoch, the system is affected if and only if the server is busy. When the server fails it is sent to repair immediately. And after repair the server is as good as new. As soon as the repair of the server is completed, the server is idle. We suppose that the server takes a vacation of random length after an exponential time with mean β^{-1} when the server is up (busy or idle). The customer just being served before server vacation leaves the service area to join the orbit. It is assumed that the service time for the interrupted customer is invalid. That is to say, the interrupted customer must restart to receive service. At the end of vacation, the server is idle and waits for the customer if any are in the orbit, or for a new customer to arrive.

All positive customers have i.i.d. service time distribution given by

$$B(x) = 1 - \exp\left\{-\int_0^x \eta(t)\mathrm{d}t\right\}$$

with mean $\mu \in (0, +\infty)$.

The vacation time distribution is given by

$$V(x) = 1 - \exp\left\{-\int_0^x \gamma(t)\mathrm{d}t\right\}$$

with mean $\nu \in (0, +\infty)$.

The repair time distribution is given by

$$R(x) = 1 - \exp\left\{-\int_0^x \varphi(t)\mathrm{d}t\right\}$$

with mean $r \in (0, +\infty)$.

We assume that all the random variables defined above are independent. Throughout the rest of the paper, we refer to "positive customers" as "customers". We denote by $\bar{F}(x) = 1 - F(x)$ the tail of distribution function $F(x)$. We also denote $\lambda = \lambda^+ + \lambda^-$, $F^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}F(x)$, $\tilde{\bar{F}}(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\bar{F}(x)\mathrm{d}x = \frac{1-F^*(s)}{s}$.

3. Stability condition

In this section, we carry out the necessary and sufficient condition for the system to be stable. According to the model description, the customer's departure from the system is due to the service completion or the removal of a negative arrival. Let $\{t_n; n \in N\}$ be the sequence of epochs of the end of the service completion times or the repair completion times at which the server is idle. We define $\widetilde{B}_n = t_n - t_{n-1}$ to be the generalized service time of the nth customer, that is, the length of time since the nth customer begins to retry until the service is completed or the repair is completed, where $\widetilde{B}_n$ includes the retrial time of the customer and a possible repair time of the server due to a negative arrival and a possible vacation time of the server due to the vacation strategy during the service period of the nth customer. It is obvious that $\widetilde{B}_n$ is independent of n, so we denote by $\widetilde{B}$ the random variable which represents the generalized service time with distribution function $\widetilde{B}(x)$.

We assume that the system is idle at time $t = 0$. In order to find the distribution $\widetilde{B}(x)$, we regard a new queueing system where the state of the tagged customer who leaves the system is assumed to be an absorbent state. In this new queueing system, we define the states of the server as

$$C(t) = \begin{cases} 0, & \text{if the customer leaves the system at time } t, \\ 1, & \text{if the server is idle at time } t, \\ 2, & \text{if the server is busy at time } t, \\ 3, & \text{if the server is on vacation at time } t, \\ 4, & \text{if the server is under repair at time } t. \end{cases}$$

For $t \ge 0$, we define the random variable $\xi(t)$ as follows:

(i) if $C(t) = 2$, $\xi(t)$ represents the elapsed service time of the customer currently being served at time t;
(ii) if $C(t) = 3$, $\xi(t)$ represents the elapsed vacation time at time t;
(iii) if $C(t) = 4$, $\xi(t)$ represents the elapsed repair time at time t.

Then, $\{\widetilde{X}(t); t \ge 0\} = \{C(t), \xi(t), t \ge 0\}$ is a Markov process and the state 0 is assumed to be an absorbent state. In this new queueing model, we define the transient probabilities of the process $\widetilde{X}(t)$ as follows:

$$P_0(t) = P\{C(t) = 0\}, \quad t \ge 0, \qquad q_0(t) = P_0'(t),$$
$$q_1(t) = P\{C(t) = 1\}, \quad t \ge 0,$$
$$q_i(t, x)\mathrm{d}x = P\{C(t) = i, x < \xi(t) \le x + \mathrm{d}x\}, \quad t \ge 0,\ x > 0, i = 2, 3, 4.$$

By the supplementary variable technique, we obtain the following equations that govern the dynamics of the system behaviour:

$$\frac{\mathrm{d}q_1(t)}{\mathrm{d}t} = -(\delta + \lambda^+ + \beta)q_1(t) + \int_0^\infty \gamma(x)q_3(t,x)\mathrm{d}x, \tag{3.1}$$

$$\frac{\partial q_2(t,x)}{\partial t} + \frac{\partial q_2(t,x)}{\partial x} = -(\beta + \lambda^- + \eta(x))q_2(t,x), \tag{3.2}$$

$$\frac{\partial q_3(t,x)}{\partial t} + \frac{\partial q_3(t,x)}{\partial x} = -\gamma(x)q_3(t,x), \tag{3.3}$$

$$\frac{\partial q_4(t,x)}{\partial t} + \frac{\partial q_4(t,x)}{\partial x} = -\varphi(x)q_4(t,x), \tag{3.4}$$

$$f(t) = \frac{\mathrm{d}q_0(t)}{\mathrm{d}t} = \int_0^\infty \eta(x)q_2(t,x)\mathrm{d}x + \int_0^\infty \varphi(x)q_4(t,x)\mathrm{d}x. \tag{3.5}$$

The boundary conditions are

$$q_2(t,0) = (\lambda^+ + \delta)q_1(t), \tag{3.6}$$

$$q_3(t,0) = \beta\left[q_1(t) + \int_0^\infty q_2(t,x)\mathrm{d}x\right], \tag{3.7}$$

$$q_4(t,0) = \lambda^- \int_0^\infty q_2(t,x)\mathrm{d}x. \tag{3.8}$$

The initial condition is

$$q_1(0) = 1. \tag{3.9}$$

Taking Laplace transforms with respect to t in Eqs. (3.1)–(3.9) yields

$$(s + \delta + \lambda^+ + \beta)\widetilde{q_1}(s) = 1 + \int_0^\infty \gamma(x)\widetilde{q_3}(s,x)\mathrm{d}x, \tag{3.10}$$

$$\frac{\partial \widetilde{q_2}(s,x)}{\mathrm{d}x} = -(s + \beta + \lambda^- + \eta(x))\widetilde{q_2}(s,x), \tag{3.11}$$

$$\frac{\partial \widetilde{q_3}(s,x)}{\mathrm{d}x} = -(s + \gamma(x))\widetilde{q_3}(s,x), \tag{3.12}$$

$$\frac{\partial \widetilde{q_4}(s,x)}{\mathrm{d}x} = -(s + \varphi(x))\widetilde{q_4}(s,x), \tag{3.13}$$

$$\widetilde{f}(s) = \int_0^\infty \eta(x)\widetilde{q_2}(s,x)\mathrm{d}x + \int_0^\infty \varphi(x)\widetilde{q_4}(s,x)\mathrm{d}x, \tag{3.14}$$

$$\widetilde{q_2}(s,0) = (\lambda^+ + \delta)\widetilde{q_1}(s), \tag{3.15}$$

$$\widetilde{q_3}(s,0) = \beta\left[\widetilde{q_1}(s) + \int_0^\infty \widetilde{q_2}(s,x)\mathrm{d}x\right], \tag{3.16}$$

$$\widetilde{q_4}(s,0) = \lambda^- \int_0^\infty \widetilde{q_2}(s,x)\mathrm{d}x. \tag{3.17}$$

Solving Eqs. (3.11)–(3.13), we get

$$\widetilde{q_2}(s,x) = \widetilde{q_2}(s,0)\exp\{-(s + \beta + \lambda^-)x\}\bar{B}(x), \tag{3.18}$$

$$\widetilde{q_3}(s,x) = \widetilde{q_3}(s,0)\exp\{-sx\}\bar{V}(x), \tag{3.19}$$

$$\widetilde{q_4}(s,x) = \widetilde{q_4}(s,0)\exp\{-sx\}\bar{R}(x). \tag{3.20}$$

Substituting Eq. (3.19) into (3.10) leads to

$$\widetilde{q_1}(s) = \frac{1 + \widetilde{q_3}(s,0)V^*(s)}{s + \delta + \lambda^+ + \beta}. \tag{3.21}$$

Combining Eqs. (3.15), (3.16) and (3.18), we obtain

$$\widetilde{q_3}(s,0) = \beta[1 + (\lambda^+ + \delta)\widetilde{\bar{B}}(s + \beta + \lambda^-)]\widetilde{q_1}(s). \tag{3.22}$$

From (3.21) and (3.22), we obtain

$$\widetilde{q_1}(s) = \frac{1}{s+\delta+\lambda^+ +\beta-\beta V^*(s)[1+(\lambda^+ +\delta)\widetilde{\widetilde{B}}(s+\beta+\lambda^-)]}. \tag{3.23}$$

Substituting Eq. (3.23) into (3.15) leads to

$$\widetilde{q_2}(s,0) = \frac{\lambda^+ +\delta}{s+\delta+\lambda^+ +\beta-\beta V^*(s)[1+(\lambda^+ +\delta)\widetilde{\widetilde{B}}(s+\beta+\lambda^-)]}. \tag{3.24}$$

Combining Eqs. (3.17), (3.18) and (3.24), we obtain

$$\widetilde{q_4}(s,0) = \frac{\lambda^-(\lambda^+ +\delta)\widetilde{\widetilde{B}}(s+\beta+\lambda^-)}{s+\delta+\lambda^+ +\beta-\beta V^*(s)[1+(\lambda^+ +\delta)\widetilde{\widetilde{B}}(s+\beta+\lambda^-)]}. \tag{3.25}$$

From (3.14), (3.18), (3.20), (3.24) and (3.25), we can obtain the following lemmas.

Lemma 3.1. *The Laplace–Stieltjes transform of $\widetilde{B}(x)$ is given by*

$$\widetilde{B}^*(s) = \widetilde{q_0}(s) = \frac{(\lambda^+ +\delta)[B^*(s+\beta+\lambda^-)+\lambda^-\widetilde{\widetilde{B}}(s+\beta+\lambda^-)R^*(s)]}{s+\delta+\lambda^+ +\beta-\beta V^*(s)[1+(\lambda^+ +\delta)\widetilde{\widetilde{B}}(s+\beta+\lambda^-)]}. \tag{3.26}$$

Lemma 3.2. *The expected value of the generalized service time is given by*

$$E\widetilde{B} = \frac{(\lambda^+ +\delta)(1+\beta\nu+\lambda^- r)[1-B^*(\beta+\lambda^-)]+(\lambda^- +\beta)(1+\beta\nu)}{(\lambda^+ +\delta)[\lambda^- +\beta B^*(\beta+\lambda^-)]}. \tag{3.27}$$

Proof. From (3.26) and the following equation

$$E\widetilde{B} = -\widetilde{q_0}'(s)|_{s=0},$$

we obtain (3.27). □

The main result of this section is the following theorem.

Theorem 3.1. *The necessary and sufficient condition for the system to be stable is*

$$\lambda^+ \frac{(\lambda^+ +\delta)(1+\beta\nu+\lambda^- r)[1-B^*(\beta+\lambda^-)]+(\lambda^- +\beta)(1+\beta\nu)}{(\lambda^+ +\delta)[\lambda^- +\beta B^*(\beta+\lambda^-)]} < 1. \tag{3.28}$$

Proof. We consider the generalized service times $\widetilde{B_n}$ as the "service times" of the nth customers, then the queueing system considered in this paper can be investigated as a classical $M/\widetilde{G}/1$ queue where the input process is a homogeneous Poisson process with rate λ^+ and the successive service times $\{\widetilde{B_n}\}$ are identically distributed random variables with distribution function $\widetilde{B}(x)$. The necessary and sufficient condition for the ordinary $M/\widetilde{G}/1$ queue to be stable is $\lambda^+(E\widetilde{B}) < 1$, so the inequality $\lambda^+(E\widetilde{B}) < 1$ is also the necessary and sufficient condition for the queueing system considered in this paper to be stable. □

4. Steady-state distribution

In this section, we study the steady-state distribution for the system under consideration. Let $N(t)$ be the orbit size (i.e. the number of customers in the retrial group) at time t. We use the same notations as in the previous section, then, $\{X(t); t \geq 0\} = \{C(t), N(t), \xi(t), t \geq 0\}$ is a Markov process.

We assume that the stability condition is fulfilled. We define the limiting probabilities

$$I_n = \lim_{t\to\infty} P\{C(t)=1, N(t)=n\}, \quad n \geq 0$$

and the limiting probability densities

$$S_n(x)\mathrm{d}x = \lim_{t\to\infty} P\{C(t)=2, N(t)=n, x<\xi(t)\leq x+\mathrm{d}x\}, \quad x\geq 0, n\geq 0,$$

$$V_n(x)\mathrm{d}x = \lim_{t\to\infty} P\{C(t)=3, N(t)=n, x<\xi(t)\leq x+\mathrm{d}x\}, \quad x\geq 0, n\geq 0,$$

$$R_n(x)\mathrm{d}x = \lim_{t\to\infty} P\{C(t)=4, N(t)=n, x<\xi(t)\leq x+\mathrm{d}x\}, \quad x\geq 0, n\geq 0.$$

4.1. The steady state equations

By the method of the supplementary variable, we easily obtain the system of equilibrium equations:

$$\frac{\mathrm{d}S_n(x)}{\mathrm{d}x} = -(\lambda+\beta+\eta(x))S_n(x) + \lambda^+ S_{n-1}(x)(1-\delta_{n,0}), \quad n \geq 0, \tag{4.1}$$

$$\frac{\mathrm{d}V_n(x)}{\mathrm{d}x} = -(\lambda^+ +\gamma(x))V_n(x) + \lambda^+ V_{n-1}(x)(1-\delta_{n,0}), \quad n \geq 0, \tag{4.2}$$

$$\frac{\mathrm{d}R_n(x)}{\mathrm{d}x} = -(\lambda^+ +\varphi(x))R_n(x) + \lambda^+ R_{n-1}(x)(1-\delta_{n,0}), \quad n \geq 0, \tag{4.3}$$

$$[\lambda^+ + \beta + \delta(1-\delta_{n,0})]I_n = \int_0^\infty S_n(x)\eta(x)\mathrm{d}x + \int_0^\infty R_n(x)\varphi(x)\mathrm{d}x + \int_0^\infty V_n(x)\gamma(x)\mathrm{d}x, \quad n \geq 0, \tag{4.4}$$

where $\delta_{n,m}$ is the Kronecker delta. The steady-state boundary conditions are

$$S_n(0) = \lambda^+ I_n + \delta I_{n+1}, \quad n \geq 0, \tag{4.5}$$

$$V_n(0) = \beta\left[\int_0^\infty S_{n-1}(x)\mathrm{d}x + I_n\right], \quad n \geq 0, \tag{4.6}$$

$$R_n(0) = \lambda^- \int_0^\infty S_n(x)\mathrm{d}x, \quad n \geq 0 \tag{4.7}$$

and the normalization condition is

$$\sum_{n=0}^\infty I_n + \sum_{n=0}^\infty \int_0^\infty S_n(x)\mathrm{d}x + \sum_{n=0}^\infty \int_0^\infty V_n(x)\mathrm{d}x + \sum_{n=0}^\infty \int_0^\infty R_n(x)\mathrm{d}x = 1. \tag{4.8}$$

In the remainder of this section, we solve Eqs. (4.1)–(4.8).

4.2. The model solution

In order to solve the system of Eqs. (4.1)–(4.7), let us define the following probability generating functions for $|z| \leq 1$:

$$I(z) = \sum_{n=0}^\infty I_n z^n, \qquad S(x,z) = \sum_{n=0}^\infty S_n(x) z^n, \qquad V(x,z) = \sum_{n=0}^\infty V_n(x) z^n, \qquad R(x,z) = \sum_{n=0}^\infty R_n(x) z^n.$$

The following theorem discusses the steady-state distribution of the system.

Theorem 4.1. *The stationary distribution of the process $\{X(t), t \geq 0\}$ has the following generating functions*

$$I(z) = \frac{\delta(z-F(z))I_0}{\beta z[1-V^*(\lambda^+ - \lambda^+ z)] + \lambda^+ z(1-F(z)) + \delta(z-F(z))}, \tag{4.9}$$

$$S(x,z) = I(z)(\lambda^+ - W(z))\exp\{-(\lambda+\beta-\lambda^+ z)x\}\bar{B}(x), \tag{4.10}$$

$$V(x,z) = \beta I(z)[(\lambda^+ - W(z))z\tilde{\bar{B}}(\lambda+\beta-\lambda^+ z) + 1]\exp\{-(\lambda^+ - \lambda^+ z)x\}\bar{V}(x), \tag{4.11}$$

$$R(x,z) = \lambda^- I(z)(\lambda^+ - W(z))\tilde{\bar{B}}(\lambda+\beta-\lambda^+ z)\exp\{-(\lambda^+ - \lambda^+ z)x\}\bar{R}(x), \tag{4.12}$$

where

$$F(z) = B^*(\lambda+\beta-\lambda^+ z) + \tilde{\bar{B}}(\lambda+\beta-\lambda^+ z)[\lambda^- R^*(\lambda^+ - \lambda^+ z) + \beta z V^*(\lambda^+ - \lambda^+ z)],$$

$$W(z) = \frac{\beta[1-V^*(\lambda^+ - \lambda^+ z)] + \lambda^+(1-F(z))}{z-F(z)},$$

and

$$I_0 = \frac{(\lambda^- + \beta)(\delta - \beta\lambda^+\nu) - (\lambda^+ + \delta)[\beta + \lambda^+(1+\lambda^- r + \beta\nu)][1-B^*(\lambda^- + \beta)]}{\delta(1+\beta\nu)[\lambda^- + \beta B^*(\lambda^- + \beta)]}.$$

Proof. Multiplying Eqs. (4.1)–(4.7) by z^n and summing over all possible values of n, we obtain the following equations:

$$\frac{\partial S(x,z)}{\partial x} = -[\lambda + \beta - \lambda^+ z + \eta(x)]S(x,z), \tag{4.13}$$

$$\frac{\partial V(x,z)}{\partial x} = -[\lambda^+(1-z) + \gamma(x)]V(x,z), \tag{4.14}$$

$$\frac{\partial R(x,z)}{\partial x} = -[\lambda^+(1-z) + \varphi(x)]R(x,z), \tag{4.15}$$

$$(\beta + \lambda^+ + \delta)I(z) = \delta I_0 + \int_0^\infty S(x,z)\eta(x)\mathrm{d}x + \int_0^\infty V(x,z)\gamma(x)\mathrm{d}x + \int_0^\infty R(x,z)\varphi(x)\mathrm{d}x, \tag{4.16}$$

$$S(0,z) = \lambda^+ I(z) + \frac{\delta}{z}(I(z) - I_0), \tag{4.17}$$

$$V(0,z) = \beta\left[z\int_0^\infty S(x,z)\mathrm{d}x + I(z)\right], \tag{4.18}$$

$$R(0,z) = \lambda^- \int_0^\infty S(x,z)\mathrm{d}x. \tag{4.19}$$

The solutions of Eqs. (4.13)–(4.15) are given by

$$S(x,z) = S(0,z)\exp\{-(\lambda + \beta - \lambda^+ z)x\}\bar{B}(x), \tag{4.20}$$

$$V(x,z) = V(0,z)\exp\{-(\lambda^+ - \lambda^+ z)x\}\bar{V}(x), \tag{4.21}$$

$$R(x,z) = R(0,z)\exp\{-(\lambda^+ - \lambda^+ z)x\}\bar{R}(x). \tag{4.22}$$

Putting (4.17) into (4.20), we get

$$S(x,z) = \left[\lambda^+ I(z) + \frac{\delta}{z}(I(z) - I_0)\right]\exp\{-(\lambda + \beta - \lambda^+ z)x\}\bar{B}(x). \tag{4.23}$$

Substituting (4.23) into (4.18) and (4.19), we get

$$V(0,z) = \beta\left\{[(\lambda^+ z + \delta)\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z) + 1]I(z) - I_0\delta\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z)\right\}, \tag{4.24}$$

$$R(0,z) = \lambda^-\left[\lambda^+ I(z) + \frac{\delta}{z}(I(z) - I_0)\right]\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z). \tag{4.25}$$

Putting (4.24) and (4.25) into (4.21) and (4.22) respectively, we obtain the required results (4.11) and (4.12). Combining (4.16) and (4.17), (4.20)–(4.22), (4.24) and (4.25) and solving for $I(z)$, after some algebraic manipulation, we get the required result (4.9). Finally, I_0 is calculated from the normalization (4.8). $\square$

Remark 4.1. Since it is necessary that $I_0 > 0$, we obtain that (3.28) is a necessary condition for the stability of the system.

Next we are interested in investigating marginal orbit size distributions due to the system state of the server.

Theorem 4.2. *Under the steady-state condition, the marginal probability generating functions of the server's state orbit size distribution are given by*

$$I(z) = \frac{\delta(z - F(z))I_0}{\beta z[1 - V^*(\lambda^+ - \lambda^+ z)] + \lambda^+ z(1 - F(z)) + \delta(z - F(z))},$$

$$S(z) = \int_0^{+\infty} S(x,z)\mathrm{d}x = I(z)(\lambda^+ - W(z))\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z),$$

$$V(z) = \int_0^{+\infty} V(x,z)\mathrm{d}x = \beta I(z)[(\lambda^+ - W(z))z\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z) + 1]\tilde{\bar{V}}(\lambda^+ - \lambda^+ z),$$

$$R(z) = \int_0^{+\infty} R(x,z)\mathrm{d}x = \lambda^- I(z)(\lambda^+ - W(z))\tilde{\bar{B}}(\lambda + \beta - \lambda^+ z)\tilde{\bar{R}}(\lambda^+ - \lambda^+ z).$$

Next the system state probabilities are given in Corollary 4.1.

Corollary 4.1. *If the system is in steady-state condition, then*

(i) *the probability that the server is idle is*

$$P_I = \lim_{z\to 1} I(z) = \frac{\lambda^- + \beta - [\beta + \lambda^+(1 + \lambda^- r + \beta\nu)][1 - B^*(\beta + \lambda^-)]}{(1 + \beta\nu)[\lambda^- + \beta B^*(\beta + \lambda^-)]}.$$

(ii) *the probability that the server is busy serving a customer is*

$$P_S = \lim_{z\to 1} S(z) = \frac{\lambda^+[1 - B^*(\beta + \lambda^-)]}{\lambda^- + \beta B^*(\beta + \lambda^-)}.$$

(iii) *the probability that the server is on vacation is*

$$P_V = \lim_{z\to 1} V(z) = \frac{\lambda^- + \beta - (\lambda^+\lambda^- r + \beta)[1 - B^*(\beta + \lambda^-)]}{(1 + \beta\nu)[\lambda^- + \beta B^*(\beta + \lambda^-)]}\beta\nu.$$

(iv) *the probability that the server is under repair is*

$$P_R = \lim_{z\to 1} R(z) = \frac{\lambda^+\lambda^- r[1 - B^*(\beta + \lambda^-)]}{\lambda^- + \beta B^*(\beta + \lambda^-)}.$$

Corollary 4.2. (1) *The probability generating function of the number of customers in the orbit is given by*

$$P_o(z) = I(z) + S(z) + V(z) + R(z) = I(z)\phi(z),$$

where

$$\phi(z) = 1 + (\lambda^+ - W(z))[1 + z\beta\tilde{\tilde{V}}(\lambda^+ - \lambda^+ z) + \lambda^-\tilde{\tilde{R}}(\lambda^+ - \lambda^+ z)]\tilde{\tilde{B}}(\lambda + \beta - \lambda^+ z) + \beta\tilde{\tilde{V}}(\lambda^+ - \lambda^+ z).$$

(2) *The probability generating function of the number of customers in the system is given by*

$$P_s(z) = I(z) + zS(z) + V(z) + R(z) = I(z)\psi(z),$$

where

$$\psi(z) = 1 + (\lambda^+ - W(z))[z + z\beta\tilde{\tilde{V}}(\lambda^+ - \lambda^+ z) + \lambda^-\tilde{\tilde{R}}(\lambda^+ - \lambda^+ z)]\tilde{\tilde{B}}(\lambda + \beta - \lambda^+ z) + \beta\tilde{\tilde{V}}(\lambda^+ - \lambda^+ z).$$

Corollary 4.3. (1) *The mean number of customers in the orbit is* $L_o = P_o'(1)$.
(2) *The mean number of customers in the system is* $L_s = P_s'(1)$.

Remark 4.2. The explicit expressions of the mean number of customers in the system or orbit are very complex and we omit the details here. Readers may compute them directly according to the specific expressions of $P_o(z)$ and $P_s(z)$. (See Section 9.)

5. Stochastic decomposition

In this section, we investigate the stochastic decomposition property of the system size distribution. This property has been studied in many vacation models. The classical stochastic decomposition property shows that the steady-state system size at an arbitrary point can be represented as the sum of two independent random variables, one of which is the system size of the corresponding queueing system without server vacations and the other is the orbit size given that the server is on vacations (see [34]). Furthermore, stochastic decomposition has also been held for retrial queues (see [4,35]). In particular, in the context of our system, we have the following theorem.

Theorem 5.1. *The number of customers in the system under study (L_s) can be expressed as the sum of two independent random variables, one of which is the number of customers in the M/G/1 G-queue with non-exhaustive random vacations and an unreliable server (L_∞) and the other is the number of customers in the orbit given that the server is idle (L). That is,* $L_s = L_\infty + L$.

Proof. We observe that the generating function of the system size distribution can be decomposed as follows:

$$P_s(z) = P_\infty(z)Q(z),$$

where

$$P_{\infty}(z) = \frac{\lambda^- + \beta - (\lambda^+\lambda^- r + \lambda^+\beta\nu + \lambda^+ + \beta)[1 - B^*(\beta + \lambda^-)]}{(1 + \beta\nu)[\lambda^- + \beta B^*(\beta + \lambda^-)]}\psi(z),$$

$$\begin{aligned} Q(z) &= \frac{z - F(z)}{\beta z[1 - V^*(\lambda^+ - \lambda^+ z)] + \lambda^+ z(1 - F(z)) + \delta(z - F(z))} \\ &\quad \times \frac{(\lambda^- + \beta)(\delta - \beta\lambda^+\nu) - (\lambda^+ + \delta)[\beta + \lambda^+(1 + \lambda^- r + \beta\nu)][1 - B^*(\lambda^- + \beta)]}{\lambda^- + \beta - (\lambda^+\lambda^- r + \lambda^+\beta\nu + \lambda^+ + \beta)[1 - B^*(\beta + \lambda^-)]} \\ &= \frac{I(z)}{I(1)}. \end{aligned}$$

We note that $P_{\infty}(z) = \lim_{\delta\to\infty} P_s(z)$ is the probability generating function of the number of customers in the $M/G/1$ G-queue with non-exhaustive random vacations and an unreliable server. $Q(z)$ is the probability generating function of the number of customers in the orbit given that the server is idle. In consequence, in terms of convolution, we can state that $L_s = L_{\infty} + L$. □

6. Reliability analysis

In this section, we discuss some reliability indexes of the system.

Suppose that the system is initially empty. Let $A(t)$ be the pointwise availability of the server at time t, that is, the probability that the server is either serving a customer or the server is on vacation or the server is idle, such that the steady state availability of the server will be $A = \lim_{t\to\infty} A(t)$.

Corollary 6.1. *The steady state availability of the server is given by*

$$A = 1 - P_R = 1 - \frac{\lambda^+\lambda^- r[1 - B^*(\beta + \lambda^-)]}{\lambda^- + \beta B^*(\beta + \lambda^-)}.$$

Corollary 6.2. *The steady state failure frequency of the server is given by*

$$W_f = \lambda^- P_S = \frac{\lambda^-\lambda^+[1 - B^*(\beta + \lambda^-)]}{\lambda^- + \beta B^*(\beta + \lambda^-)}.$$

Denote by τ the time to the first failure of the server, then the reliability function of the server is

$$\varsigma(t) = P(\tau > t).$$

In order to find the reliability of the server, letting the failure states of the server be absorbing states, then we obtain a new system. In the new system, we use the same notations as in the previous section, then we can get the following differential equations:

$$\frac{\partial S_n(x,t)}{\partial x} + \frac{\partial S_n(x,t)}{\partial t} = -(\lambda + \beta + \eta(x))S_n(x,t) + \lambda^+ S_{n-1}(x,t)(1 - \delta_{n,0}), \quad n \ge 0, \tag{6.1}$$

$$\frac{\partial V_n(x,t)}{\partial x} + \frac{\partial V_n(x,t)}{\partial t} = -(\lambda^+ + \gamma(x))V_n(x,t) + \lambda^+ V_{n-1}(x,t)(1 - \delta_{n,0}), \quad n \ge 0, \tag{6.2}$$

$$\frac{\mathrm{d}I_n(t)}{\mathrm{d}t} = -[\lambda^+ + \beta + \delta(1 - \delta_{n,0})]I_n(t) + \int_0^{\infty} S_n(x,t)\eta(x)\mathrm{d}x + \int_0^{\infty} V_n(x,t)\gamma(x)\mathrm{d}x, \quad n \ge 0, \tag{6.3}$$

the boundary conditions:

$$S_n(0,t) = \lambda^+ I_n(t) + \delta I_{n+1}(t), \quad n \ge 0, \tag{6.4}$$

$$V_n(0,t) = \beta\left[\int_0^{\infty} S_{n-1}(x,t)\mathrm{d}x + I_n(t)\right], \quad n \ge 0, \tag{6.5}$$

and the initial conditions:

$$I_n(0) = \delta_{n,0}, \qquad S_n(x,0) = 0, \qquad V_n(x,0) = 0.$$

By taking Laplace transforms of Eqs. (6.1)–(6.5), we obtain

$$\frac{\partial \widetilde{S}_n(x,s)}{\partial x} = -(s+\lambda+\beta+\eta(x))\widetilde{S}_n(x,s) + \lambda^+\widetilde{S}_{n-1}(x,s)(1-\delta_{n,0}), \quad n \geq 0, \tag{6.6}$$

$$\frac{\partial \widetilde{V}_n(x,s)}{\partial x} = -(s+\lambda^+)\widetilde{V}_n(x,s) + \lambda^+\widetilde{V}_{n-1}(x,s)(1-\delta_{n,0}), \quad n \geq 0, \tag{6.7}$$

$$s\widetilde{I}_n(s) = \delta_{n,0} - [\lambda^+ + \beta + \delta(1-\delta_{n,0})]\widetilde{I}_n(s) + \int_0^\infty \widetilde{S}_n(x,s)\eta(x)\mathrm{d}x + \int_0^\infty \widetilde{V}_n(x,s)\gamma(x)\mathrm{d}x, \quad n \geq 0, \tag{6.8}$$

$$\widetilde{S}_n(0,s) = \lambda^+\widetilde{I}_n(s) + \delta\widetilde{I}_{n+1}(s), \quad n \geq 0, \tag{6.9}$$

$$\widetilde{V}_n(0,s) = \beta\left[\int_0^\infty \widetilde{S}_{n-1}(x,s)\mathrm{d}x + \widetilde{I}_n(s)\right], \quad n \geq 0. \tag{6.10}$$

Define the following generating functions

$$\widetilde{I}(z,s) = \sum_{n=0}^\infty \widetilde{I}_n(s)z^n, \qquad \widetilde{S}(z,x,s) = \sum_{n=0}^\infty \widetilde{S}_n(x,s)z^n, \qquad \widetilde{V}(z,x,s) = \sum_{n=0}^\infty \widetilde{V}_n(x,s)z^n.$$

Then multiplying Eqs. (6.6)–(6.10) by z^n and summing over all possible values of n, we obtain the following equations:

$$\frac{\partial \widetilde{S}(z,x,s)}{\partial x} = -(s+\lambda+\beta+\eta(x)-\lambda^+z)\widetilde{S}(z,x,s), \tag{6.11}$$

$$\frac{\partial \widetilde{V}(z,x,s)}{\partial x} = -(s+\lambda^+ + \gamma(x)-\lambda^+z)\widetilde{V}(z,x,s), \tag{6.12}$$

$$(s+\lambda^+ + \beta+\delta)\widetilde{I}(z,s) = 1 + \delta\widetilde{I}_0(s) + \int_0^\infty \widetilde{S}(z,x,s)\eta(x)\mathrm{d}x + \int_0^\infty \widetilde{V}(z,x,s)\gamma(x)\mathrm{d}x, \tag{6.13}$$

$$z\widetilde{S}(z,0,s) = (\lambda^+z+\delta)\widetilde{I}(z,s) - \delta\widetilde{I}_0(s), \tag{6.14}$$

$$\widetilde{V}(z,0,s) = \beta\left[z\int_0^\infty \widetilde{S}(z,x,s)\mathrm{d}x + \widetilde{I}(z,s)\right]. \tag{6.15}$$

The solutions of the partial differential equation (6.11) and (6.12) are given by

$$\widetilde{S}(z,x,s) = \widetilde{S}(z,0,s)\exp\{-(s+\lambda+\beta-\lambda^+z)x\}\bar{B}(x), \tag{6.16}$$

$$\widetilde{V}(z,x,s) = \widetilde{V}(z,0,s)\exp\{-(s+\lambda^+-\lambda^+z)x\}\bar{V}(x). \tag{6.17}$$

Substituting (6.14) into (6.16), we obtain

$$\widetilde{S}(z,x,s) = \frac{1}{z}[(\lambda^+z+\delta)\widetilde{I}(z,s) - \delta\widetilde{I}_0(s)]\exp\{-(s+\lambda+\beta-\lambda^+z)x\}\bar{B}(x). \tag{6.18}$$

Combining (6.15), (6.17) and (6.18), we get

$$\widetilde{V}(z,x,s) = \beta\{[(\lambda^+z+\delta)\widetilde{I}(z,s) - \delta\widetilde{I}_0(s)]\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z) + \widetilde{I}(z,s)\}\exp\{-(s+\lambda^+-\lambda^+z)x\}\bar{V}(x). \tag{6.19}$$

Putting (6.18) and (6.19) into (6.13), we obtain

$$M(z,s)\widetilde{I}(z,s) = N(z,s)\delta\widetilde{I}_0(s) - z, \tag{6.20}$$

where

$$\begin{aligned} M(z,s) \;=\; & (\lambda^+z+\delta)[B^*(s+\lambda+\beta-\lambda^+z) + z\beta\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)V^*(s+\lambda^+-\lambda^+z)] \\ & + z\beta V^*(s+\lambda^+-\lambda^+z) - z(s+\lambda^+ + \beta+\delta), \end{aligned}$$

$$N(z,s) = B^*(s+\lambda+\beta-\lambda^+z) + z\beta\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)V^*(s+\lambda^+-\lambda^+z) - z.$$

It is easy to see that

(i) $M(0,s) = \delta B^*(s+\lambda+\beta) \geq 0$,

(ii) $M(1,s) = -(\lambda^+ + \delta)[1 - B^*(s+\lambda^- + \beta) - \beta V^*(s)\widetilde{\bar{B}}(s+\lambda^- + \beta)] - \beta[1-V^*(s)] - s < 0$,

(iii) $\frac{\partial^2}{\partial z^2}M(z,s) \geq 0$, i.e., function $M(z,s)$ is a convex function for each fixed s.

Hence, for each fixed s the equation $M(z,s)=0$ has exactly a root $z=f(s)$ in the interval [0, 1].
Choosing $z=f(s)$ in (6.20) yields

$$\widetilde{I}_0(s)=\frac{f(s)}{\delta N(f(s),s)}. \tag{6.21}$$

Substituting (6.21) into (6.20), we get

$$\widetilde{I}(z,s)=\frac{f(s)N(z,s)-zN(f(s),s)}{M(z,s)N(f(s),s)}. \tag{6.22}$$

Substituting (6.21) and (6.22) into (6.18) and (6.19), we get

$$\widetilde{S}(z,x,s)=\frac{(\lambda^+z+\delta)[f(s)N(z,s)-zN(f(s),s)]-M(z,s)f(s)}{zM(z,s)N(f(s),s)}\exp\{-(s+\lambda+\beta-\lambda^+z)x\}\bar{B}(x), \tag{6.23}$$

$$\begin{aligned}\widetilde{V}(z,x,s) &= \frac{[(\lambda^+z+\delta)\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)+1][f(s)N(z,s)-zN(f(s),s)]-M(z,s)f(s)\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)}{M(z,s)N(f(s),s)}\\ &\quad\times\exp\{-(s+\lambda^+-\lambda^+z)x\}\beta\bar{V}(x).\end{aligned} \tag{6.24}$$

So we summarize our results in the following theorem.

Theorem 6.1. *The Laplace transform of $\varsigma(t)$ is given by*

$$\begin{aligned}\widetilde{\varsigma}(s) &= \frac{f(s)N(1,s)-N(f(s),s)}{M(1,s)N(f(s),s)}\{1+(\lambda^++\delta)\widetilde{\bar{B}}(s+\lambda^-+\beta)+[(\lambda^++\delta)\widetilde{\bar{B}}(s+\lambda^-+\beta)+1]\beta\widetilde{\bar{V}}(s)\}\\ &\quad-\frac{f(s)}{N(f(s),s)}\widetilde{\bar{B}}(s+\lambda^-+\beta)[1+\beta\widetilde{\bar{V}}(s)].\end{aligned} \tag{6.25}$$

Proof. From (6.23) and (6.24) we can obtain

$$\widetilde{S}(z,s)=\int_0^\infty\widetilde{S}(z,x,s)\mathrm{d}x=\frac{(\lambda^+z+\delta)[f(s)N(z,s)-zN(f(s),s)]-M(z,s)f(s)}{zM(z,s)N(f(s),s)}\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z),$$

$$\begin{aligned}\widetilde{V}(z,s) &= \int_0^\infty\widetilde{V}(z,x,s)\mathrm{d}x\\ &= \beta\left\{\frac{[(\lambda^+z+\delta)\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)+1][f(s)N(z,s)-zN(f(s),s)]}{M(z,s)N(f(s),s)}\right\}\widetilde{\bar{V}}(s+\lambda^+-\lambda^+z)\\ &\quad-\frac{f(s)\beta\widetilde{\bar{B}}(s+\lambda+\beta-\lambda^+z)\widetilde{\bar{V}}(s+\lambda^+-\lambda^+z)}{N(f(s),s)}.\end{aligned}$$

Hence we have

$$\widetilde{\varsigma}(s)=\widetilde{I}(1,s)+\widetilde{S}(1,s)+\widetilde{V}(1,s).$$

By the direct calculation we can obtain (6.25). □

From Theorem 6.1 we can obtain the following result.

Corollary 6.3. *The mean time to the first failure (MTTFF) of the server is given by*

$$\mathrm{MTTFF}=\frac{f(0)N(1,0)-N(f(0),0)}{M(1,0)N(f(0),0)}[1+(\lambda^++\delta)\widetilde{\bar{B}}(\lambda^-+\beta)](1+\beta\nu)-\frac{f(0)\widetilde{\bar{B}}(\lambda^-+\beta)(1+\beta\nu)}{N(f(0),0)}, \tag{6.26}$$

where

$$M(1,0)=-(\lambda^++\delta)[1-B^*(\lambda^-+\beta)-\beta\widetilde{\bar{B}}(\lambda^-+\beta)],\qquad N(1,0)=B^*(\lambda^-+\beta)+\beta\widetilde{\bar{B}}(\lambda^-+\beta)-1,$$
$$N(f(0),0)=B^*(\lambda+\beta-\lambda^+f(0))+f(0)\beta\widetilde{\bar{B}}(\lambda+\beta-\lambda^+f(0))V^*(\lambda^+-\lambda^+f(0))-f(0),$$

where $f(0)$ is the unique root of z of the equation

$$(\lambda^+z+\delta)[B^*(\lambda+\beta-\lambda^+z)+z\beta\widetilde{\bar{B}}(\lambda+\beta-\lambda^+z)V^*(\lambda^+-\lambda^+z)]+z\beta V^*(\lambda^+-\lambda^+z)-z(\lambda^++\beta+\delta)=0$$

in the interval [0, 1].

Proof. From (6.25) and the following equation

$$\mathrm{MTTFF} = \int_0^{+\infty} \varsigma(t)\mathrm{d}t = \widetilde{\varsigma}(s)|_{s=0},$$

we obtain (6.26). □

7. Busy period

This section is dedicated to the investigation of the busy period of our model. The busy period is defined as the time period which starts when a external customer arrives into an empty system and ends at the next departure epoch when the system is empty. We denote by D the random variable which represents the length of the busy period, $D(x)$ its distribution function and $D^*(s)$ its Laplace–Stieltjes transform. It is clear that the busy period of our queueing system agrees with the busy period of the $M/\widetilde{G}/1$ queue where the service times are considered as the "generalized service times". So we have the following theorem.

Theorem 7.1. *The Laplace–Stieltjes transform of the busy period satisfies the equation*

$$D^*(s) = \widetilde{q_0}(s + \lambda^+(1 - D^*(s))).$$

The mean length of the busy period is

$$E[D] = \frac{(\lambda^+ + \delta)(1 + \beta\nu + \lambda^- r)[1 - B^*(\beta + \lambda^-)] + (\lambda^- + \beta)(1 + \beta\nu)}{(\lambda^+ + \delta)\{\lambda^- + \beta B^*(\beta + \lambda^-) - \lambda^+(1 + \beta\nu + \lambda^- r)[1 - B^*(\beta + \lambda^-)]\} - \lambda^+(\lambda^- + \beta)(1 + \beta\nu)}. \tag{7.1}$$

Proof. According to the Takacs equation of the ordinary $M/\widetilde{G}/1$ queue, we have

$$D^*(s) = \widetilde{B}^*(s + \lambda^+(1 - D^*(s))) = \widetilde{q_0}(s + \lambda^+(1 - D^*(s))).$$

By differentiating the above equation, we have

$$E[D] = \frac{1}{\frac{1}{E\widetilde{B}} - \lambda^+}.$$

By using (3.27) and on simplification, we can get (7.1). □

8. Special cases

Case 1. No vacation

In this special case, our model becomes the $M/G/1$ retrial G-queue with an unreliable server.

We put $\beta = 0$ in the main results and obtain the following.

The necessary and sufficient condition for the system to be stable is

$$\lambda^+(1 + \lambda^- r)\frac{1 - B^*(\lambda^-)}{\lambda^-}\frac{\delta + \lambda^+}{\delta} < 1.$$

The marginal probability generating functions of the server's state orbit size distribution are given by

$$I(z) = \frac{\delta(z - F(z))I_0}{(\lambda^+ + \delta)z - (\lambda^+ z + \delta)F(z)},$$

$$S(z) = \lambda^+\frac{z - 1}{z - F(z)}I(z)\widetilde{\widetilde{B}}(\lambda - \lambda^+ z),$$

$$R(z) = \lambda^-\lambda^+\frac{z - 1}{z - F(z)}I(z)\widetilde{\widetilde{B}}(\lambda - \lambda^+ z)\widetilde{\widetilde{R}}(\lambda^+ - \lambda^+ z),$$

where

$$F(z) = B^*(\lambda - \lambda^+ z) + \lambda^-\widetilde{\widetilde{B}}(\lambda - \lambda^+ z)R^*(\lambda^+ - \lambda^+ z)$$

and

$$I_0 = 1 - \lambda^+(1 + \lambda^- r)\frac{1 - B^*(\lambda^-)}{\lambda^-}\frac{\delta + \lambda^+}{\delta}.$$

Table 1
Performance measures for varying values of λ^- with $(\delta, \beta) = (5, 0.5)$.

λ^-	P_I	P_S	P_V	P_R	A	W_f
0.1	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328
0.5	0.5330	0.3077	0.1401	0.0192	0.9808	0.1538
1.0	0.5408	0.2857	0.1378	0.0357	0.9643	0.2857
5.0	0.5779	0.1818	0.1266	0.1136	0.8864	0.9091
10	0.5982	0.1250	0.1205	0.1563	0.8438	1.2500

Table 2
Performance measures for varying values of β with $(\lambda^-, \delta) = (0.1, 5)$.

β	P_I	P_S	P_V	P_R	A	W_f
0.1	0.6359	0.3279	0.0321	0.0041	0.9959	0.0328
0.3	0.5775	0.3279	0.0905	0.0041	0.9959	0.0328
0.6	0.5020	0.3279	0.1660	0.0041	0.9959	0.0328
0.9	0.4382	0.3279	0.2298	0.0041	0.9959	0.0328
1.2	0.3835	0.3279	0.2845	0.0041	0.9959	0.0328

Remark 8.1. It should be pointed out that our system is assumed that a negative customer not only removes a positive customer from the queue but also causes the server breakdown. So these results are not consistent with known results in [9] which the server breakdowns were driven by another independent Poisson process and the customer just being served before server failure waits for the server to complete his remaining service. But we note that the stable condition $\lambda^+\left[(1+\lambda^- r)\frac{1-B^*(\lambda^-)}{\lambda^-}+\frac{1}{\lambda^++\delta}\right] < 1$ agrees with the stable condition which was investigated in [9] if we regard the $\lambda^+\frac{1-B^*(\lambda^-)}{\lambda^-} = \rho$ as the load of the system. In fact, $\frac{1-B^*(\lambda^-)}{\lambda^-} \leq \mu$ and $\lim_{\lambda^-\to 0}\frac{1-B^*(\lambda^-)}{\lambda^-} = \mu$, since arrival of a negative customer removes the customer being in service.

Case 2. No negative customers, no vacation

In this special case, our model becomes the $M/G/1$ retrial queue with constant rate of repeated attempts.

We assume that $(\lambda^-, \beta) \to (0, 0)$ in the main results and obtain:

The necessary and sufficient condition for the system to be stable is

$$\lambda^+\mu\frac{\delta+\lambda^+}{\delta} < 1.$$

The marginal probability generating functions of the server's state orbit size distribution are given by

$$I(z) = \frac{\delta[z - B^*(\lambda^+ - \lambda^+ z)]I_0}{(\lambda^+ + \delta)z - (\lambda^+ z + \delta)B^*(\lambda^+ - \lambda^+ z)},$$
$$S(z) = \lambda^+\frac{z-1}{z - B^*(\lambda^+ - \lambda^+ z)}I(z)\widetilde{\widetilde{B}}(\lambda^+ - \lambda^+ z),$$

where

$$I_0 = 1 - \lambda^+\mu\frac{\delta+\lambda^+}{\delta}.$$

It may be noted that the results above agree with known results of the special cases in [6,9,10].

9. Application to cellular mobile networks and numerical examples

In this section, we give an application for the model under discussion and some numerical results as follows. We consider a single cell of a cellular mobile network that consists of one base station and one channel which provides exponential service times of rate μ_b. The fresh calls arrive at the channel according to a Poisson process with rate λ^+ and a fresh call is blocked if it finds the channel busy. Any blocked fresh call joins the orbit and retries for service after an exponentially distributed time with rate δ/n, given that there are n calls in the orbit. In addition, the channel may interrupt the service after an exponentially distributed time with rate β due to the inhibiting signal. Furthermore, sometimes the channel gets attacked by a virus and the call in the channel is lost forever (we assume the virus arrival rate is λ^-). Then the channel can be modelled as an $M/G/1$ retrial G-queue with random vacations and an unreliable server. For the purpose of a numerical illustration, we assume that all distribution functions in this paper are exponential, i.e. $B(x)$, $V(x)$, $R(x)$ are exponential distribution functions and their parameters are μ_b, μ_v, μ_r respectively. Here we choose the following arbitrary values: $\mu_b = 1/\mu = 6$, $\mu_v = 1/\nu = 3$, $\mu_r = 1/r = 8$, $\lambda^+ = 2$. Of course, in all the cases below, the parametric values are chosen in such a way that the stability condition holds. Numerical results are reported in Tables 1–3 and Figs. 1–4.

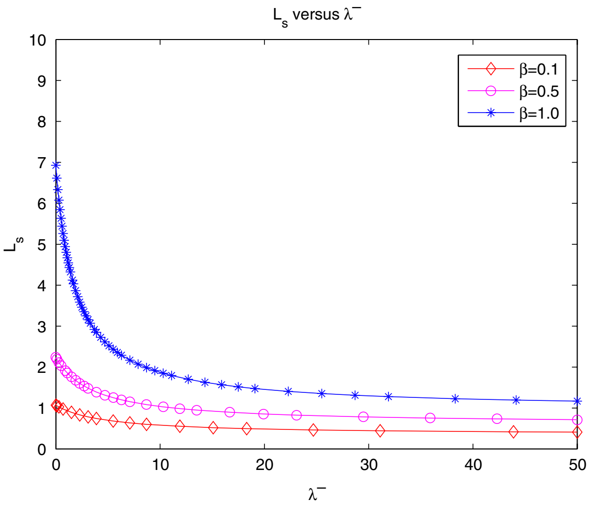

Fig. 1. Mean number of customers in the system versus λ^-.

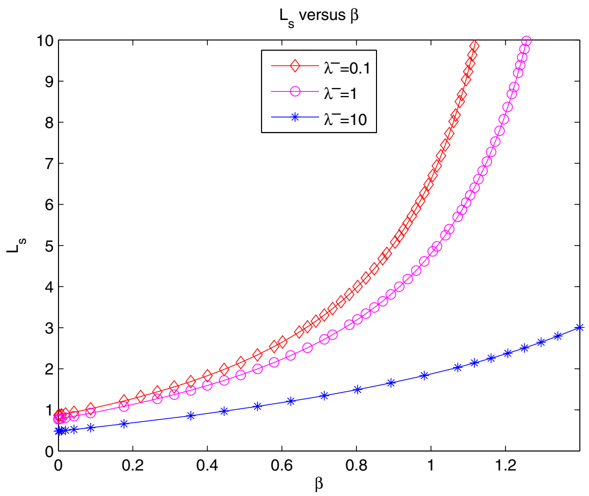

Fig. 2. Mean number of customers in the system versus β.

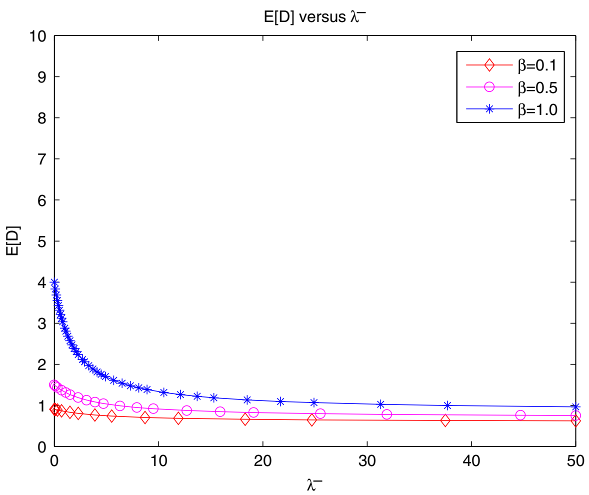

Fig. 3. Mean length of the busy period versus λ^-.

Table 3
Performance measures for varying values of δ with $(\lambda^-, \beta) = (0.1, 0.5)$.

δ	P_I	P_S	P_V	P_R	A	W_f
3	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328
5	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328
10	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328
50	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328
100	0.5258	0.3279	0.1423	0.0041	0.9959	0.0328

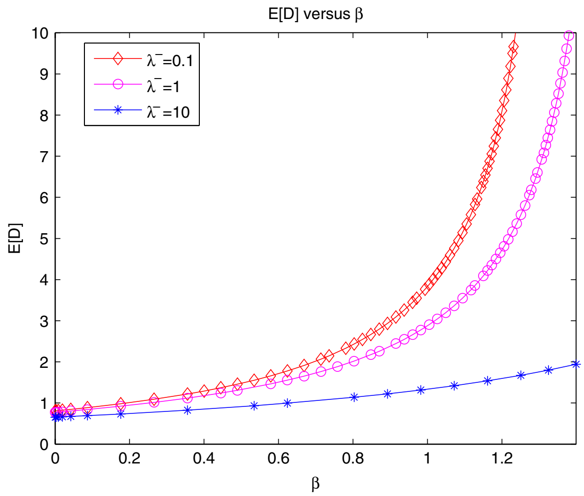

Fig. 4. Mean length of the busy period versus β.

Table 1 shows that the effects of the rate of negative arrivals on the performance measures of the system for the set of parameters $(\delta, \beta) = (5, 0.5)$. We observe that the probabilities P_S, P_V decrease monotonously and the probability P_R increases monotonously as the value λ^- increases. This is because the negative arrivals not only remove the customer being in service but also cause the server breakdown and when the server is down, it does not take vacation. The effects of β on the performance measures of the system are reported in Table 2, where we set $(\lambda^-, \delta) = (0.1, 5)$. As is to be expected, the probability P_V increases monotonously as the value β increases. But it is interesting that the value β does not affect the probabilities P_S and P_R. The effect of varying retrial rate δ on the performance measures of the system is shown in Table 3 for the set of parameters $(\lambda^-, \beta) = (0.1, 0.5)$. We observe that the retrial rate δ does not affect the performance measures of the system, which is very interesting.

In Figs. 1 and 2, the mean number of customers in the system is plotted against the arrival rate λ^- of negative customers and the vacation rate β, respectively. As is to be expected, L_s decreases monotonously as the value λ^- increases. L_s increases monotonously as the value β increases. Moreover, we observe that, as λ^- increases to infinity, L_s tends to a constant because the limitation $\lim_{\lambda^- \to \infty} L_s$ exists. For example, when $\beta = 0.5$, we have $\lim_{\lambda^- \to \infty} L_s = 0.6071$. We would like to remark that, as β approaches the stability condition, L_s tends to infinity because the system becomes unstable. The same discussion holds for the expected value of the busy period in Figs. 3 and 4. The trends shown by the figures are as expected.

10. Conclusion

In this paper we have analysed a single server retrial queue with negative customers and non-exhaustive random vacations subject to the server breakdowns and repairs. Essentially, a breakdown is represented by a negative customer arriving at the server which removes the customer being in service when the server is busy. The server takes a vacation of random length after an exponential time when the server is up. For the constant rate of repeated attempts, the necessary and sufficient condition for the system stability is analysed by finding absorb distribution and using the stable condition of the classical $M/G/1$ queueing system. The joint distribution of the server state and the orbit size in the steady-state are discussed. Analytical expressions for various performance measures of interest, the reliability analysis and the busy period are studied. The stochastic decomposition is investigated. An application for the model under discussion is provided and numerical examples have been carried out to observe the influence of the main parametric values. The most interesting aspect of this paper is that we established a Markov chain with an absorbent state to obtain the Laplace–Stieltjes transform of the "generalized service time" distribution. It is clear that our queueing system can be studied as a classical $M/G/1$ queueing system where the service times are considered as the "generalized service times". It is proved that our method to obtain the stability conditions of complicated queueing systems is efficient computationally and is tractable and feasible for most examples.

Acknowledgments

This research is partially supported by the Postdoctoral Foundation of Central South University. The authors would like to thank the anonymous referees for their valuable comments and suggestions which have definitely helped to improve the presentation of this paper.

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